On Sums of Remainders and Almost Perfect Numbers

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In this note we state a conjecture concerning the sum of remainders $S(n)$ of a natural number n on division by all $m<n$. If true, this conjecture would imply that the powers of two are the only almost perfect numbers.

Note. This note is dedicated to Prof. H.A. Lauwerier on occasion of his 65th birthday.

For every problem you can't
solve, there exists an easier
problem you also can't solve.
Find it.
(P. Erdös)

"Nothing works!"
(Catweazle)

Thanks to Euclid it is known that for any two natural numbers n,m $(m \neq 0)$ there exist unique natural numbers k,r with $n = m \cdot k + r$, $0 \leqslant r < m$. Let us write $\mathrm{mod}(n,m) = r$ for the remainder on division by m. We now define the following number theoretic function,

$$S(n) := \sum_{i<n} \mathrm{mod}(n,i) \qquad n \in \mathbb{N},\ i \in \mathbb{N}_{>0}.$$

This function was probably first mentioned by Lucas [4]. His result on $S(n)$, which appears here as (6) gives an immediate link of $S(n)$ to perfect (and related) numbers (see also [1]).

For convenience we introduce $\rho(n,i) := i \cdot \lfloor n/i \rfloor$ (with $\lfloor x \rfloor$ being the largest integer $\leqslant x$). It is easily seen that

$$n = \rho(n,i) + \mathrm{mod}(n,i). \tag{1}$$

Summing both sides of (1) over all $i \leqslant n$ yields, after some rearranging,

$$S(n) = n^2 - \sum_{i=1}^{n} \rho(n,i). \tag{2}$$

As the last term is positive, we have $S(n) < n^2$ for all n. A smaller upper bound is obtained after summing (1) over the non-divisors of n

$$S(n) = n^2 - n \cdot d(n) - \sum_{i \nmid n} \rho(n,i),$$

where $d(n) := \#\{k \leqslant n : k \mid n\}$. This yields $S(n) \leqslant n^2 - n \cdot d(n)$, with equality only for $n \leqslant 2$. As a bound this is not very useful, because $d(n)$ is a function even more erratic than $S(n)$. However, we can derive a limit-relation between $d(n)$ and $S(n)$. For this we need a few well-known results. The first is by Catalan ([2])

$$\sigma(k) = \sum_{i=1}^{k} (\rho(k,i) - \rho(k-1,i)), \tag{3}$$

where $\sigma(k) := \Sigma_{i|k} i$. Combining (3) with (2) gives (Cesàro)

$$S(n) = n^2 - \sum_{k=1}^{n} \sigma(k). \tag{4}$$

Using a result from Hardy & Wright ([3], Theorem 324) we can rewrite this as

$$S(n) = n^2 - \tfrac{1}{12}\pi^2 n^2 + O(n \log n), \text{ as } n \to \infty.$$

This yields an estimate for $S(n)$: $S(n) \sim (1 - \frac{1}{12}\pi^2) \cdot n^2$. Since $d(n)/n \to 0$ as $n \to \infty$ ([3]), we have

PROPOSITION. *Let $n \in \mathbb{N}$, then*

$$\lim_{n \to \infty} \frac{n^2 - n \cdot d(n)}{S(n)} = \frac{1}{1 - \frac{1}{12}\pi^2}.$$

An obvious question is: Can one characterize pairs (n,m) of natural numbers with $S(n) = S(m)$? Concerning this we state the following

CONJECTURE. *Let $n,m \in \mathbb{N}$, $n \neq m$, then*

$$S(n) = S(m) \Leftrightarrow n = 2^k,\ m = 2^k - 1 \text{ for some } k \geqslant 0. \tag{5}$$

We will show that the implication from right to left holds. In fact we will show something stronger. The proof rests on the following

THEOREM (LUCAS). *Let $n \in \mathbb{N}_{>0}$, then*

$$S(n) - S(n-1) = 2n - 1 - \sigma(n). \tag{6}$$

The Theorem follows immediately after subtracting $S(n-1)$ from $S(n)$, both written as in equation (4). If in (6) we take n to be a prime-power, we get

$$S(p^n) - S(p^n - 1) = \frac{p-2}{p-1}(p^n - 1).$$

The implication '$\Leftarrow$' of (5) now follows from the observation that for $n > 0$: $S(p^n) = S(p^n - 1) \Leftrightarrow p = 2$.

As already stated, (6) connects $S(n)$ with almost perfect, perfect and quasi-perfect numbers. Recall that a number n is called perfect if $\sigma(n) = 2n$, almost perfect if $\sigma(n)=2n-1$, and quasiperfect if $\sigma(n)=2n+1$. If we put $P(n):=S(n)-S(n-1)$ then: $P(n)=0 \Leftrightarrow n$ is almost perfect; $P(n)=-1 \Leftrightarrow n$ is perfect; $P(n)=-2 \Leftrightarrow n$ is quasiperfect.

It is known at least since Euler that all even perfect numbers are of the form $2^{n-1}(2^n-1)$ with 2^n-1 a (Mersenne) prime. It is not known whether there are any odd perfect numbers. There are no known quasiperfect numbers and the only known almost perfect numbers are the powers of two.

If our Conjecture is true, it would follow that the powers of two are indeed the only almost perfect numbers. This relation to the somewhat esoteric problem of 'perfection of numbers' makes the existence of an elementary proof of the left-to-right implication very unlikely.

REMARK. Further study of the functions $S(n)$ and $P(n)$ might be helpfull. For example, taking some fixed $n \in \mathbb{N}$ and calculating $\text{mod}(n, 1)$, $\text{mod}(n, 2)$, ..., $\text{mod}(n, n)$, we notice that $S(n)$ may be considered as being composed of subsums, each of which is the contribution to $S(n)$ of a set $\{k, k+1, \ldots, k+l\}$ of numbers $\leqslant n$, with the property that $\exists m \in \mathbb{N}$ such that for all $0 \leqslant i < l$, $\text{mod}(n, k+i) - \text{mod}(n, k+i+1) = m$. In fact, for each $m \in \mathbb{N}$ there is a unique set with this property (most of these will be empty), and sets belonging to different m's are disjoint. So for each $m \in \mathbb{N}$ we find a subsum, (which will in most cases be zero), and $S(n)$ can be expressed as the infinite sum of these subsums. This implies of course that $P(n)$ can be considered as being composed of related subsums. The calculation of these subsums shows strikingly regular patterns.

In a more detailed technical note (available on request) we have derived closed expressions for these subsums.

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